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Solving the Two-Body Problem by Geometric Construction

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Since geometric analogs exist for algebraic relations that define the solution to the two-body problem, all of the normalized orbital and dynamical elements in the two-body problem can be determined with geometric construction without numerical computation. The use of double-precision computer drafting programs will generally mean no loss of numerical accuracy compared with digital computer solutions. The normalized velocity hodograph and position ellipse can be combined conveniently on the same graph. The method is illustrated for an initial orbit specified by velocity components or by the semimajor and semiminor axes. Normalized time is also computed as a function of true anomaly using geometric constructions. The standard two-body equations can be derived by using the geometric constructions in an algebra manipulation program such as MACSYMA.

Introduction

SEVERAL authors,[1-4] when solving the two-body problem, have discussed velocity hodographs, which allow a graphical determination of the velocity vector as a function of true anomaly. The velocity hodograph is a circle, a geometric construction whose function is similar to the auxiliary circle used in geometrically computing the eccentric anomaly from the true anomaly.[5,6] The two-body conic orbit also can be geometrically constructed to define the position as a function of true anomaly. A convenient graphical solution would involve plotting both the two-body orbit and the velocity hodograph in the same normalized coordinate space. The relative location of the hodograph and the conic orbit (herein assumed to be an ellipse) can be selected so that the origin of the hodograph coincides with the occupied focus of the ellipse. The same true anomaly line will then intersect both the velocity hodograph and the ellipse, simultaneously determining both position and velocity. One of the advantages of this plotting procedure is that corresponding points on two curves are readily identified. For instance,

1) The semilatus rectum point on the ellipse corresponds to the maximum radial velocity on the hodograph.

2) The end of the semiminor axis point on the ellipse corresponds to the maximum flight-path angle on the hodograph.

In addition, the true anomaly can be determined readily by specifying any of the dynamical elements.

The term "graphical solution" frequently refers to computer-generated graphs, which, depending on the number of curves in the figure, provide a feel for certain aspects of the problem or fairly exact solutions to the problem by interpolating between the curves. A "graphical solution" refers to solving the problem by interpolating between the curves. A "geometric solution" refers to solving the problem by geometric constructions with usually just a ruler and compass. A protractor is required if an angle is specified as an input or required as an output. This type of solution is possible if there are geometric analogs to the algebraic relations that define the solution of the two-body problem. The velocity hodograph and auxiliary circle are well-known geometric analogs for computing the velocity vector and the eccentric anomaly, respectively. This paper shows that similar geometric analogs involving only ruler and compass constructions can be applied to obtain all the two-body orbital and dynamical elements (i.e., those depending on the position in orbit).

Solving the two-body problem by geometric construction means that, instead of using numbers in equations, the numbers representing the initial conditions are used directly on a graph in connection with geometric constructions involving straight lines and conic curves to generate points on the graph representing the desired quantities. In effect, normal mathematical operations such as multiplication, division, trigonometry, and taking powers and roots can be replaced with geometric constructions. Most of these constructions are easy to derive using proportional triangles. Some, if not all, of these constructions can be found under the mathematical subject of "ruler and compass constructions."[7,8] The simplicity with which many mathematical operations can be performed by using geometry suggests incorporating the geometric computation of algebraic functions into elementary mathematics. Geometric solutions represent a different approach to solving problems than by ordinary numerical methods, perhaps providing some additional insight not readily apparent from the analytical solution.

Some of the more common mathematical operations involving three variables that can be derived using proportional triangles in two-dimensional Cartesian space are summarized in Fig. 1. This figure can be easily extended to involve many other mathematical functions of the three variables. Taking the nth power of a number can be accomplished by constructing a sawtoothed curve, as indicated in Fig. 2. The nth root of a number can be obtained by an iterative construction, where the sawtoothed curve to the right of the specified number must be obtained by guessing the nth root and using the sawtoothed constraint to divide the area into exactly n areas. The square root of a number can be found without iteration,[9,10] as shown in Fig. 3a. Projective geometry[11] introduces geometric analogs using numbers on only one axis, as shown in Figs. 3b–3d (these constructions generally require one extra line compared with the two-axis constructions). Figures 2 and 3 are drawn for numbers less than 1; similar figures can be drawn for numbers greater than 1.

The geometric solution to the two-body problem involves a series of steps using ruler and compass constructions based on the initial conditions. Each step has an associated algebraic representation, which will yield the two-body equations of motion when combined. Although this would be a laborious way of deriving the equations of motion if carried out by hand, the use of algebra manipulation programs such as MACSYMA

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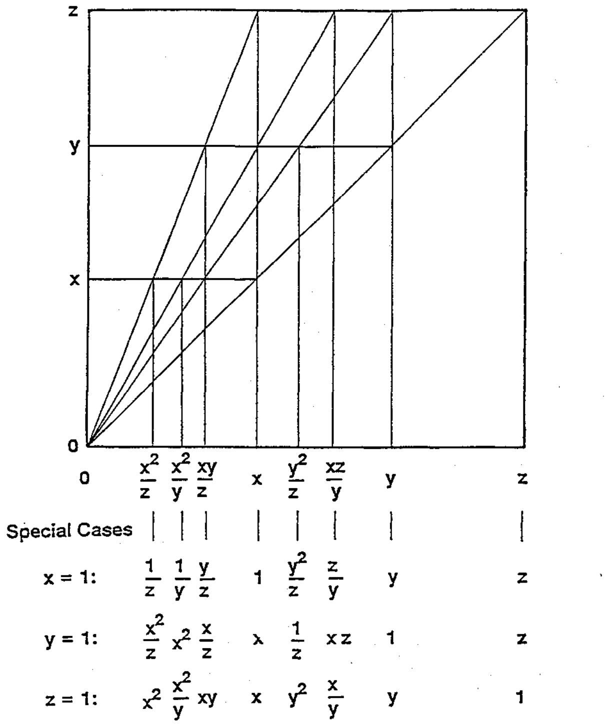

Fig. 1 Geometric calculation of three variable algebraic operations using proportional triangles.

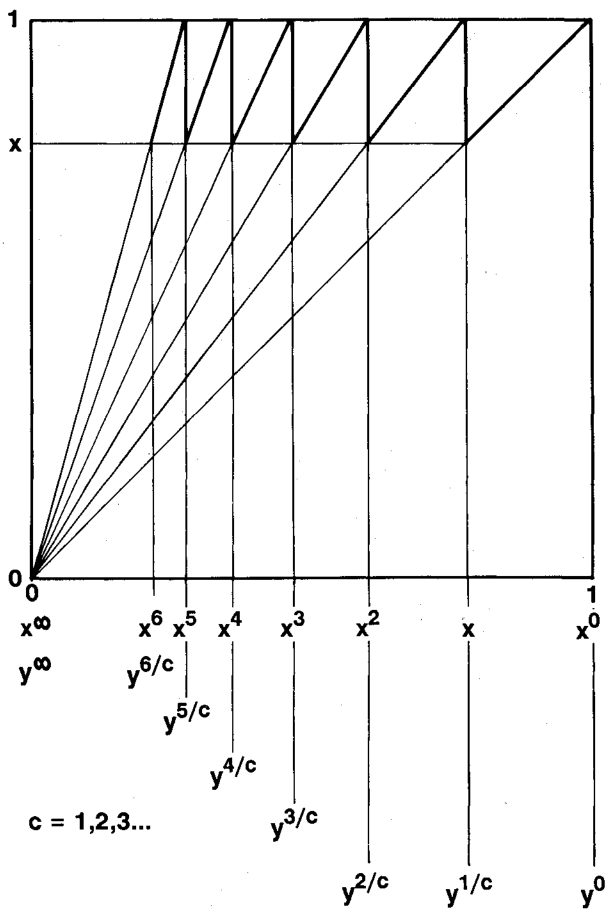

Fig. 2 Sawtoothed power curve based on proportional triangles.

might make the generation of the corresponding equations a trivial operation.

The accuracy of the geometric solution depends only on the accuracy of the construction. When constructed manually, ruler and compass constructions have limited accuracy. Computer drafting programs (e.g., IBM's CADAM program) can be used to obtain more accurate results. If double-precision computer drafting programs are used, in general, there should be no loss of numerical accuracy compared with digital computer solutions. Whether there may be some accuracy problems with some computer drafting programs and certain parabolic, rectilinear, or hyperbolic orbits is not known. For near-circular orbits, the parameter of true anomaly should be changed to the central angle.

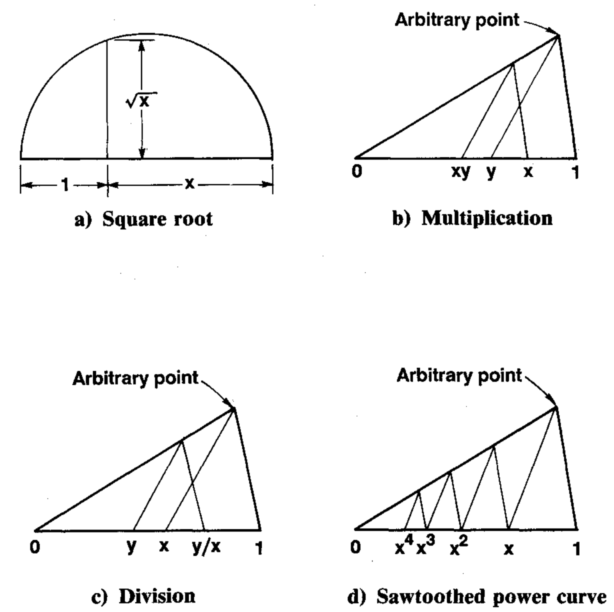

Fig. 3 Geometric analogs to algebraic functions.

Velocity Hodograph

A velocity hodograph can be introduced based on the standard two-body equations for the radial and horizontal (i.e., perpendicular to the radial direction) components of velocity normalized by dividing by the circular velocity v_{ci} at the initial point. The normalized velocity coordinates are

$$v_r = e \sin f / \beta \tag{1a}$$

$$v_h = (1 + e \cos f)/\beta \tag{1b}$$

where

$v_{ci} = \sqrt{\mu / r_i}$
v_r = radial velocity/v_{ci}
v_h = horizontal velocity/v_{ci}
e = eccentricity, $= \sqrt{\alpha^2\beta^2 + (1-\beta^2)^2}$
f = true anomaly
α = initial radial velocity/v_{ci}
β = initial horizontal velocity/v_{ci}

This hodograph will be centered at the point $(1/\beta, 0)$ and have a radius equal to e/β. Therefore, the velocity vector can always be obtained by vectorially adding a component of length $1/\beta$ to one of length e/β.

The hodograph previously used in the literature is also normalized by dividing by the circular velocity but, in addition, is multiplied by β, resulting in the following hodograph coordinates:

$$v_r\beta = e \sin f \tag{2a}$$

$$v_h\beta = 1 + e \cos f \tag{2b}$$

This hodograph circle is centered at the point (1,0) with radius equal to e. The disadvantage of this hodograph is that when unnormalizing, the values read from the graph must be multiplied by the circular velocity and divided by β.

Figure 4 shows the old and new velocity hodographs plotted in the same coordinate space. There is a proportionality factor of β between the two hodographs. The velocity vectors for the two hodographs are collinear.

Two-Body Equations of Motion

The two-body orbit in normalized form is determined by specifying two parameters. Many different combinations of parameters can be used, with perhaps the most meaningful being the specification of the initial velocity components or the semimajor and semiminor axes. In the former case, the major problem is to determine the orbital ellipse; in the latter case, the velocity has to be determined. Although the geometric construction of the two-body problem will be presented here for both cases, the two-body parameters will be specified only as functions of the initial velocity components α and β. The orbital and dynamical elements for the elliptical two-body problem—where distances are normalized by dividing by the initial radius, and velocities are normalized by dividing by the initial circular velocity—are listed in Table 1. The dynamical elements that are a function of true anomaly f are listed in Table 2. The equations in Tables 1 and 2 can be re-expressed so that the parameters are functions of the semimajor and semiminor axes.

Constructions Based on Velocity Inputs

Given the initial normalized radial α and horizontal β velocity components and a normalized initial distance of 1, the following geometric constructions (see Fig. 5) will lead to the hodograph and ellipse plotted in the x,y space (x represents the horizontal axis aligned with the semimajor axis of the ellipse) and the determination of the dynamical elements as a function of true anomaly f:

1) Construct the initial velocity vector v_i from the origin to point (β,α).

2) Construct a "radial" line (one that extends to the origin) through point $(\beta,1)$ and find where it intersects line $y = \beta$, obtaining β^2 ($=p$, semilatus rectum) on the x axis, and line $x = 1$, obtaining $1/\beta$ on the y axis.

3) Find the $x = \beta^2$ point on the initial velocity vector and measure eccentricity e as the distance to point (1,0).

4) Construct a velocity hodograph (i.e., a circle) with the origin at point $(1/\beta, 0)$ through point (β,α).

5) Divide distance $1/\beta$ in half and, using a circular arc of radius $1/2\beta$ centered at the point $(1/2\beta, 0)$, find the intersection with the hodograph, obtaining the maximum flight-path angle point, γ_m.

6) Draw a vertical line through the γ_m point. Where it intercepts the $y = \beta$ line, construct a radial line.

7) Where this radial line intercepts the line $x = 1$ defines the semimajor axis a.

8) Construct a line of length a from point $(1/\beta, 0)$ through the maximum γ point, dropping perpendicular from the end of the line to the x axis, obtaining the center of the ellipse point and the semiminor axis b.

9) Construct an ellipse based on a and b with the major axis along the x axis (see the Appendix).

10) Measure the remaining orbital elements as shown in Fig. 6 and construct the true anomaly line for the desired f, and measure the quantities r, v_r, v_h, v, and γ based on where the line intersects the hodograph and ellipse.

There are other geometric means of arriving at the same results. The complete four-quadrant solution is shown in Fig. 7. The direction of the motion is counterclockwise since the initial conditions ($\alpha = 0.2$, $\beta = 0.8$) imply heading toward apoapsis, with periapsis on the right-hand side of the figure.

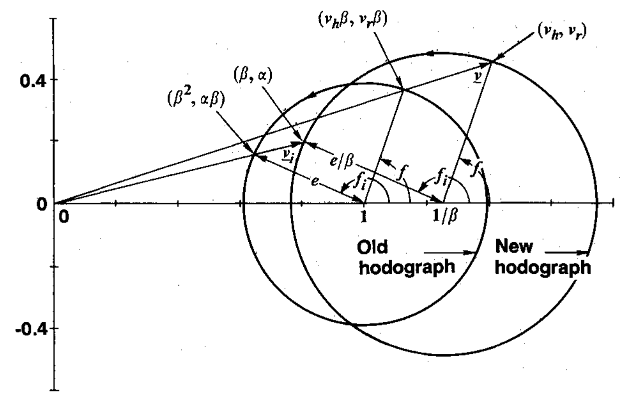

Fig. 4 Old and new hodographs.

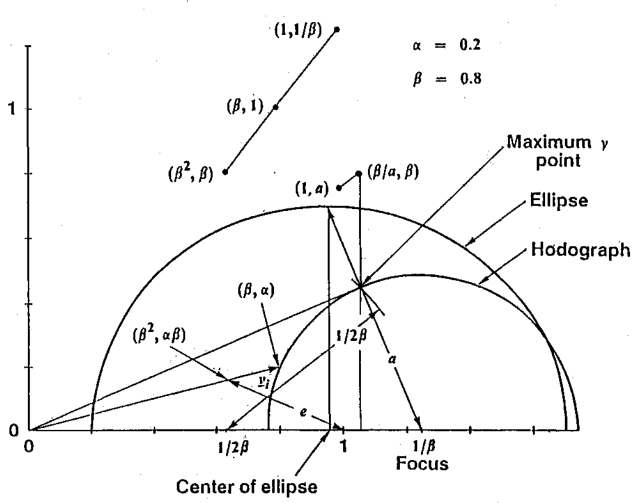

Fig. 5 Geometric construction of ellipse and hodograph based on inputs α,β.

Constructions Based on Axes Inputs

Given the normalized semimajor a and semiminor b axes, the following geometric constructions (see Fig. 8) will lead to

Table 1 Orbital elements expressed as a function of α and β

v_i	$= \sqrt{\alpha^2 + \beta^2}$	= initial velocity
γ_i	$= \tan^{-1}(\alpha/\beta)$	= initial gamma with respect to horizontal
p	$= \beta^2$	= semilatus rectum
a	$= 1/(2 - v_i^2)$	= semimajor axis
e	$= \sqrt{1 - (p/a)}$	= eccentricity
f_i	$= \cos^{-1}(p - 1)/e$	= initial true anomaly
c	$= ae$	= linear eccentricity
b	$= \sqrt{a^2 - c^2}$	= semiminor axis
r_a	$= p/(1 - e)$	= apoapsis distance
r_p	$= p/(1 + e)$	= periapsis distance
v_a	$= (1 - e)/\beta$	= apoapsis velocity
v_p	$= (1 + e)/\beta$	= periapsis velocity
v_{rm}	$= e/\beta$	= maximum radial velocity
γ_m	$= \tan^{-1}(ae/b)$	= maximum gamma
$f_{\gamma m}$	$= \cos^{-1}(-e)$	= corresponding value of true anomaly
$v_{r\gamma m}$	$= be/a\beta$	= corresponding value of radial velocity
$v_{h\gamma m}$	$= \beta/a$	= corresponding value of horizontal velocity
$v_{\gamma m}$	$= \sqrt{v_{r\gamma m}^2 + v_{h\gamma m}^2}$	= corresponding value of velocity

Table 2 Orbital elements that are a function of true anomaly

$r = p/(1 + e\cos f)$	=	radial distance
$v_r = e\sin(f/\beta)$	=	radial component of velocity
$v_h = \beta/r$	=	horizontal component of velocity
$v = \sqrt{v_r^2 + v_h^2}$	=	velocity magnitude
$\gamma = \tan^{-1}(v_r/v_h)$	=	flight-path angle with respect to horizon

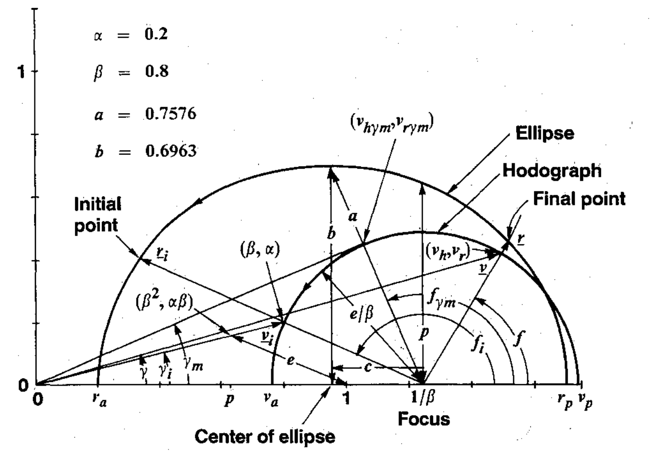

Fig. 6 Orbital and dynamical parameters.

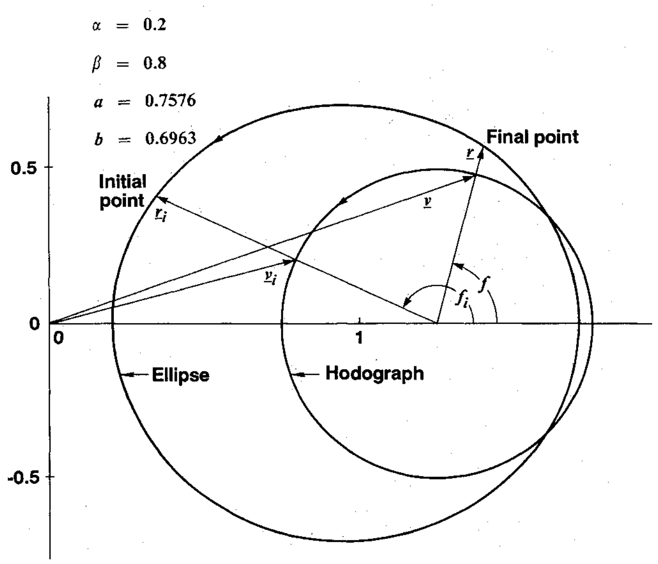

Fig. 7 Four-quadrant solution.

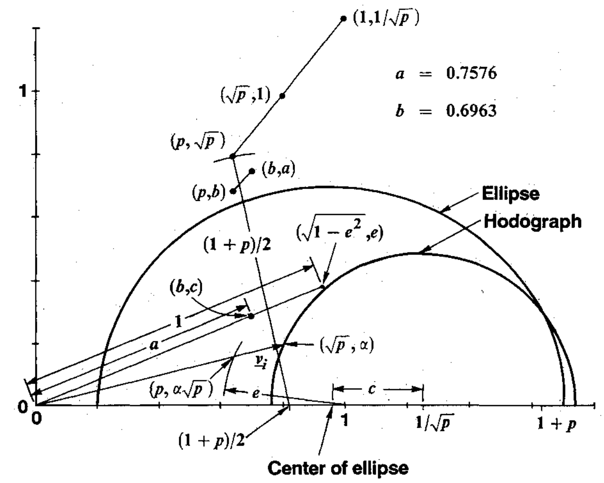

Fig. 8 Geometric construction of ellipse and hodograph based on inputs a, b.

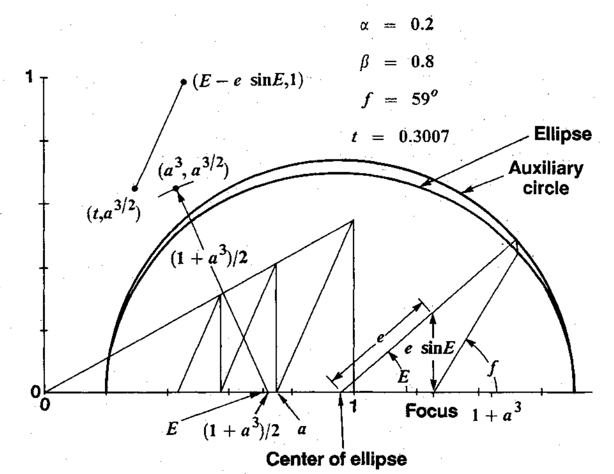

Fig. 9 Geometric construction of time.

the same hodograph and ellipse plotted with the corresponding specified velocity components:

1) Construct a radial line (i.e., a line that passes through the origin) through point (b,a) and find where it intersects line $y = b$, obtaining p.

2) Determine the square root of p by finding points $1 + p$ and then $(1 + p)/2$ on the x axis, and then swing an arc about the latter point of length $(1 + p)/2$ until it intersects the line $x = p$, establishing $\sqrt{p}$.

3) Draw a radial line through the point $(p,\sqrt{p})$ and find the intersections with $y = 1$ and $x = 1$, obtaining point $(\sqrt{p}, 1)$ and distance $1/\sqrt{p}$.

4) Swing the circular arc of radius a with the origin at (0,0) until the x component equals b, obtaining c.

5) Construct a radial line of length 1 through point (b,c), obtaining e.

6) Locate the center of an ellipse at point $(1/\sqrt{p} - c,0)$ and construct an ellipse with a and b specified (see the Appendix).

7) Swing the circular arc of radius e with the origin at (1,0) and find the intersection with line $x = p$.

8) Construct the initial velocity vector by connecting this intersection with point (0,0) and extending the line to intercept the line $x = \sqrt{p}$, obtaining radial velocity α.

9) Construct a velocity hodograph through point $(\sqrt{p},\alpha)$, using $(1/\sqrt{p},0)$ as the origin.

Measure the orbital and dynamical elements as shown in Fig. 6.

Geometric Determination of Time

The normalized time t as a function of true anomaly for elliptical orbits can be obtained by geometric construction employing similar ruler and compass constructions as used in determining the eccentric anomaly E from the true anomaly f. Time is normalized by dividing by $\sqrt{r_i^3/\mu}$, resulting in

$$t = a^{3/2}(E - e \sin E) \tag{3}$$

where E is

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan\frac{f}{2} \tag{4}$$

The geometric construction of time as shown in Fig. 9 involves the following steps:

1) Construct an auxiliary circle with its center at the center of the ellipse and radius of the semimajor axis a of the ellipse.

2) Use the true anomaly f to find a point on the ellipse and extend the vertical line through this point, finding the intersection with the auxiliary circle.

3) Construct a line between the intersection and the center of the ellipse and measure the eccentric anomaly E.

4) Measure the eccentricity e along the line and obtain $e \sin E$, the vertical component.

5) Locate E (in radians) on the x axis and subtract $e \sin E$.

6) Locate a on the x axis and obtain a^3 using the sawtoothed power curve.

7) Determine the square root of a^3 by finding points $1 + a^3$ and then $(1 + a^3)/2$ on the x axis, and then swing an arc about

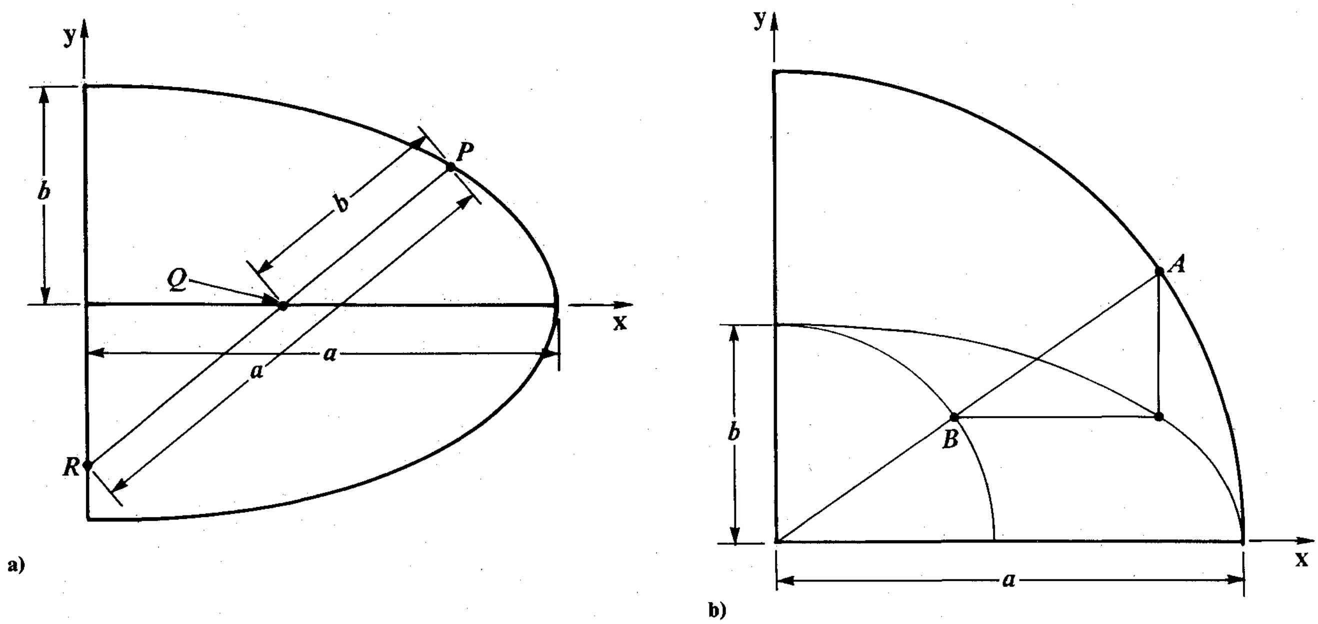

Fig. A1 Geometric constructions of an ellipse.

the latter point of length $(1 + a^3)/2$ until it intersects the line $x = a^3$, obtaining $a^{3/2}$.

8) Construct a vertical line $x = E - e \sin E$ and find its intersection with line $y = 1$.

9) From the intersection, construct a line through the origin. The intersection of this line with $y = a^{3/2}$ determines the time corresponding to the specified true anomaly.

Being able to find the time as a function of true anomaly by geometric methods means that if the time is specified (Kepler's problem) the location in the orbit can be found by geometric iteration using an initial guess of the true anomaly.

Conclusion

The use of modern computer drafting and algebra manipulation programs has introduced the possibility of solving problems by geometric construction with no loss of numerical accuracy. The geometric construction of the two-body position ellipse and the velocity hodograph as proposed here provides a quick means for computing all of the orbital and dynamical elements, independent of numerical calculation. The geometric solution can be implemented in equal or less time than associated with programming the two-body equations of motion and plotting the numerical results. One advantage of the geometric solution is that specific cutoff conditions for trajectory updates can be indicated on the computer screen, bypassing the need for special equations that terminate the orbit on those conditions. Examples of these conditions are velocity magnitude, flight-path angle, central angle, radial distance, and radial and horizontal components of velocity. If a computer drafting program with memory is used, then solutions can be obtained by just inputting the initial conditions.

Appendix: Geometric Construction of an Ellipse

The most familiar mechanical construction of an ellipse involves fastening the ends of a string of length $2a$ to two pins a distance of $2c$ apart (i.e., one pin at each focus of the ellipse), and then pulling the string taut with a moving pencil point to generate the ellipse. A method involving only a ruler is to mark on the ruler lines PR and PQ of lengths a and b. Then, while points R and Q move along the y and x axes, respectively, point P will trace out the ellipse (see Fig. A1a). A more elaborate method is to construct two concentric circles of radii a and b. Draw a line that intercepts the larger circle at A and the smaller at B. Then draw a line through A parallel to the y axis and a line through B parallel to the x axis. The locus of the intersections of these lines determines an ellipse (see Fig. A1b).

Acknowledgment

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